LPQP for MAP: Putting LP Solvers to Better Use

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MAP Inference for General Pairwise Energies

Energy Minimization

Discrete pairwise energy minimization for graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

$$\min_{\mathbf{y}} \sum_{i \in \mathcal{V}} \theta_i(y_i) + \sum_{(i,j) \in \mathcal{E}} \theta_{ij}(y_i, y_j) \qquad y_i \in \{1, \ldots, K\}.$$

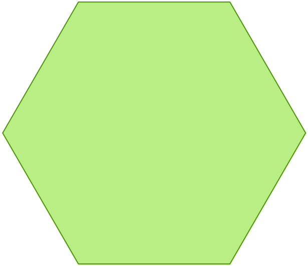
For general $\mathcal{G}$ and θ this is an NP-hard problem.

Linear and Quadratic Programming Relaxations

LP for Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{M}_\mathcal{G}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^\mathsf{T} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^\mathsf{T} \boldsymbol{\mu}_{ij}$$

But: $\mathcal{M}_\mathcal{G}$ is exponentially large!



Linear and Quadratic Programming Relaxations

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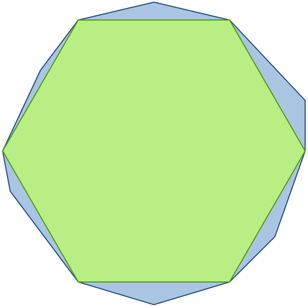
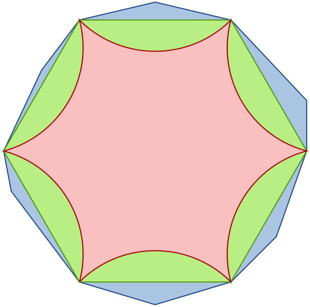
outer approximation

LP for Local Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}$$



Linear and Quadratic Programming Relaxations



LP for Marginal Polytope

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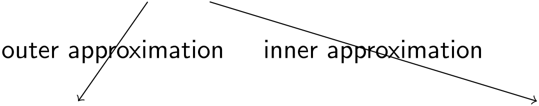

outer approximation

inner approximation

LP for Local Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_\mathcal{G}} \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^\mathsf{T} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^\mathsf{T} \boldsymbol{\mu}_{ij}$$

Quadratic Programming

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_\mathcal{G}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^\mathsf{T} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^\mathsf{T} \boldsymbol{\mu}_{ij}$$

$$\text{s.t.} \quad \boldsymbol{\mu}_{ij} = \boldsymbol{\mu}_i \boldsymbol{\mu}_j^\mathsf{T} \quad \forall (i,j) \in \mathcal{E}$$

LPQP for MAP Inference

LPQP: Best of Both Worlds?

Combine Linear and Quadratic Programming relaxations.

Joint LP and QP Objective

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{\mu} + \rho g(\boldsymbol{\mu}).$$

- $g(\boldsymbol{\mu})$: Penalty term that discourages discrepancy between

$$\boldsymbol{\mu}_{ij} \Leftrightarrow \boldsymbol{\mu}_i \boldsymbol{\mu}_j^{\mathsf{T}}$$

- Constraint set: the local marginal polytope $\mathcal{L}_{\mathcal{G}}$
- Non-convex
- Limit cases w.r.t ρ. LP relaxation: $\rho = 0$. QP relaxation: $\rho \rightarrow \infty$.

KL-Divergence Penalty Terms

Two Variants Of The Penalty Term

Uniform Weighting Of Edges:

$$\sum_{(i,j)\in\mathcal{E}} D_{KL}(\boldsymbol{\mu}_{ij}, \boldsymbol{\mu}_i \boldsymbol{\mu}_j^{\mathsf{T}})$$

Tree-based Weighting with $\mathcal{G}_a = (\mathcal{V}_a, \mathcal{E}_a)$:

$$\sum_{a\in\mathcal{A}} \eta_a \left(\sum_{(i,j)\in\mathcal{E}_a} D_{KL}(\boldsymbol{\mu}_{ij}, \boldsymbol{\mu}_i \boldsymbol{\mu}_j^{\mathsf{T}}) \right)$$

* D_{KL} denotes the Kullback-Leibler divergence

LPQP Algorithm Overview

```
Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}), \boldsymbol{\theta}$.
initialize $\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}$ uniform, $\rho = \rho_0$.
repeat
    $t = 0, \boldsymbol{\mu}^0 = \boldsymbol{\mu}$.
    repeat
        $\boldsymbol{\mu}^{t+1} = \operatorname*{argmin}_{\boldsymbol{\tau} \in \mathcal{L}_{\mathcal{G}}} u_\rho(\boldsymbol{\tau}) - \boldsymbol{\tau}^{\mathsf{T}} \nabla v_\rho(\boldsymbol{\mu}^t)$.
        $t = t + 1$.
    until $\|\boldsymbol{\mu}^t - \boldsymbol{\mu}^{t-1}\|_2 \leq \epsilon_{\mathrm{dc}}$.
    $\boldsymbol{\mu} = \boldsymbol{\mu}^t$.
    increase $\rho$.
until $\|\boldsymbol{\mu} - \boldsymbol{\mu}^0\|_2 \leq \epsilon_\rho$.
return $\boldsymbol{\mu}$.
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CCCP For a Given ρ

- Non-convex:
  apply concave-convex procedure.

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    $\boxed{\text{increase } \rho.}$
until $\|\boldsymbol{\mu} - \boldsymbol{\mu}^0\|_2 \leq \epsilon_\rho$.
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```

Gradual Increment of ρ

- Gradually moving from an LP solution to a QP solution.

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    increase $\rho$.
until $\|\boldsymbol{\mu} - \boldsymbol{\mu}^0\|_2 \leq \epsilon_\rho$.
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```

Convex Sub-Problems

Have the form:

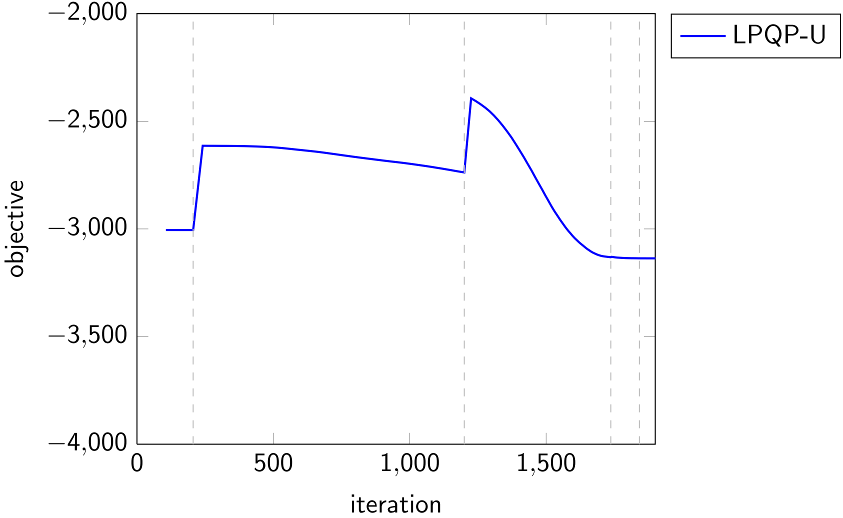
$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \mathrm{LP}(\boldsymbol{\mu})^1 - \rho \cdot \mathrm{Entropy}(\boldsymbol{\mu})^2$$

① LP with modified unary potentials → $\tilde{\boldsymbol{\theta}}$

② Entropy term, different between the penalty variants

Solved efficiently with message passing algorithms

A Run of LPQP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objective
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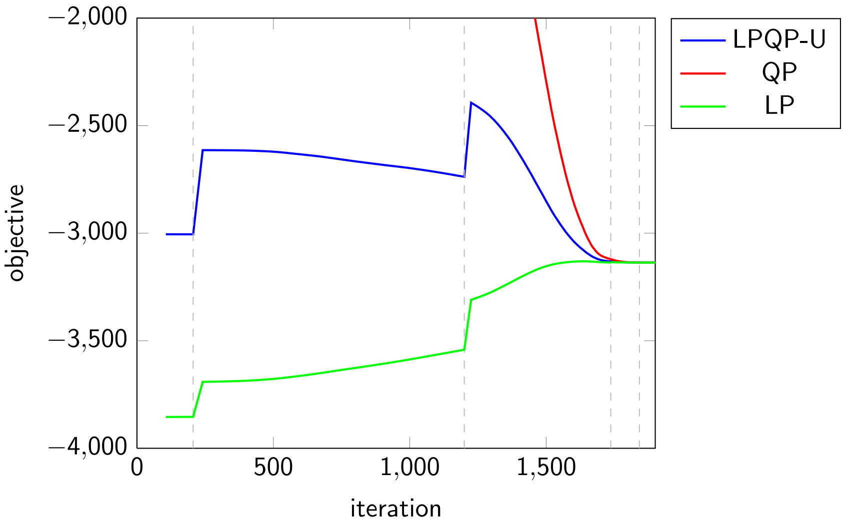
iteration
LPQP-U

A Run of LPQP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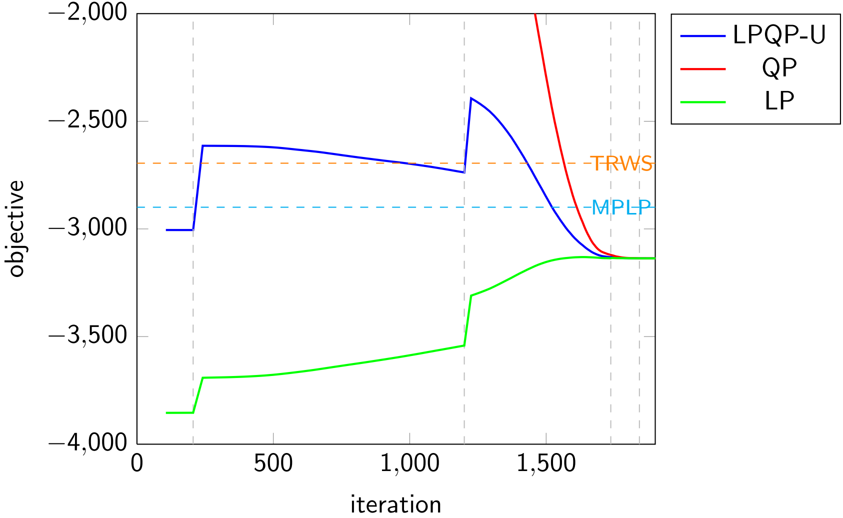
iteration
LPQP-U
QP
LP

A Run of LPQP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objective
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TRWS
MPLP
LPQP-U
QP
LP